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THE ANALYTICAL CHAOTIC MODEL OF STOCK PRICES AND ITS APPLICATIONS

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Abstract

We propose an analytical model on the chaotic behavior of stock prices. Based on the logistic equation and the Melnikov's method, we have shown circumstances under which stock prices exhibit chaotic behavior. We illustrate that the prices of Chinese listed stocks are more likely to fall into chaos due to the features of Chinese stock markets. Our theoretical predictions are consistent with the empirical results showing the multifractal properties of Chinese stock prices.

1. Introduction

The linear normal form and the normality assumption have been the foundation for contemporary financial theories and practices for decades. It is essential to the efficient-market hypothesis (EMH). EMH has made

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considerable headway and obtained a series of valuable results [5-7]. The linear normal forms think that every action produces a reaction and the reaction is direct proportion to the action. However, a group of prior studies have provided evidence that the distribution functions of price changes have fat tails and high peak, and financial markets react in a nonlinear fashion [8, 10, 19, 20, 22]. For example, Friedman et al. [10] point out that, the variation of stock prices does not match the normal distribution nicely, since there are too many extreme movements. Consequently, many studies have turned to nonlinear dynamic systems, such as chaos theory, to describe market data [9, 18, 26, 27, 29, 33]. Inconsistent with the EMH, Peters [28] shows empirical results that the standard and poor 500 index exhibits a long memory, which is an important characteristic of fractal time series. The evidence indicates that, information is not immediately impounded in stock prices, and short-term stock returns may be predictable by using technical analysis. This is to the contrast of Hsieh [17], who shows strong evidence that no chaotic state exists in U.S. stock market. The debate remains unresolved as researchers examine capital markets around the world in search of chaotic characteristics [1-3, 13, 25].

This paper contributes to the literature by bridging the gap between prior empirical results and the existing nonlinear dynamic models. We are the first to present an analytical model that systematically predicts circumstances, in which stock prices exhibit chaotic behavior. Following the coherent market hypothesis proposed by Vaga, we recognize that a market can be chaotic, random-walk like, or in transition, but it cannot stay in any state permanently [31]. Specifically, market participants may react to information in a linear fashion under some conditions, and such state may last for certain periods. However, when other conditions are satisfied, market reactions can become nonlinear and a chaotic state may emerge. Based on our model, we demonstrate that the behavior of Chinese stock prices is more likely to be chaotic.

2. Logistic Model

We demonstrate the chaotic characteristics of stock prices under the following three assumptions.

Assumption 1. All information related to supply and demand of capital such as macroeconomic, political, financial, natural, and psychological factors is reflected in the market. In other words, the above factors are captured by changes in supply and demand, which decide the stock price. In contrast to the EMH, market reactions can be linear or non-linear.

Assumption 2. There are stochastic perturbations in the stock market. The perturbations can be periodic or sporadic.

Assumption 3. The price change in the stock markets over time is a deterministic function of price, demand, and supply.

Let the price of a stock be x, the demand of the stock be N_d, and the supply be N_s. If $N_d > N_s$, then the excessive capital, which could have bought the stock if there were more supply, is $y = (N_d - N_s)x$. Choosing (x, y) as independent variables, we have the following differential equations to describe the system

$$\frac{dx}{dt} = f_1(x, y) + \varepsilon_1 g_1(x, y, t), \tag{1.1}$$

$$\frac{dy}{dt} = f_2(x, y) + \varepsilon_2 g_2(x, y, t), \tag{1.2}$$

where ε_1 and ε_2 are higher-order small quantities, t is time, and $g_1(x, y, t)$ and $g_2(x, y, t)$ are perturbations.

If $N_s > N_d$, the stock's supply surplus is $N = N_s - N_d$. Choosing (x, N) as independent variables, we have

$$\frac{dx}{dt} = f_3(x, N) + \varepsilon_3 g_3(x, N, t), \tag{2.1}$$

$$\frac{dN}{dt} = f_4(x, N) + \varepsilon_4 g_4(x, N, t), \tag{2.2}$$

where ε_3 and ε_4 are higher-order small quantities, and $g_3(x, N, t)$ and $g_4(x, N, t)$ are perturbations.

Recall that independent variables $x(t)$ and $N(t)$ are functions of time t. Suppose $g_3(x, N, t)$ and $g_4(x, N, t)$ are periodic functions, in which the period is T for both functions. Let τ be the integral multiple of T, we have

$$\overline{g_3(x, N, t)} = \frac{1}{\tau}\int_0^\tau g_3(x, N, t)\,dt = 0, \quad \overline{g_4(x, N, t)} = \frac{1}{\tau}\int_0^\tau g_4(x, N, t)\,dt = 0_\circ.$$

Defining

$$x \equiv \frac{1}{\tau}\int_0^\tau x(t)\,dt, \quad N \equiv \frac{1}{\tau}\int_0^\tau N(t)\,dt,$$

we have

$$\frac{dx}{dt} = -kNx.$$

Considering

$$kN \approx (kN)_0 + \frac{\partial}{\partial x}(kN)_{x=0}\,x,$$

we obtain

$$\frac{dx}{dt} = -(kN)_0 x - \frac{\partial}{\partial x}(kN)_{x=0}\,x^2 = Ax - Bx^2,$$

where $$A = -(kN)_0,\ B = \frac{\partial}{\partial x}(kN)_{x=0}.$$

Suppose the stock price is x_n when $t = n\tau$, and x_{n+1} when $t = (n+1)\tau$. Let the unit of time be τ, then $\Delta t = (n+1)\tau - n\tau = \tau = 1$. Thus, we have

$$x_{n+1} = x_n + \frac{dx_n}{dt}\Delta t = (A+1)x_n - Bx_n^2.$$

Suppose $\mu \equiv A + 1$, it can be shown

$$x_{n+1} = \mu x_n (1 - \frac{B}{A+1} x_n).$$

Let $u_n = \frac{B}{A+1} x_n$ and $u_{n+1} = \frac{B}{A+1} x_{n+1}$, and we obtain

$$u_{n+1} = \mu u_n (1 - u_n). \tag{3}$$

This is a typical logistic equation. Prior literature has documented that, when $\mu > \mu_\infty (\mu_\infty = 3.569945672 \cdots)$, u and x show chaotic characteristics [12].

Theorem 1. *Chaos is not an inevitable characteristic of the stock price. However, the stock price can be chaotic, when certain conditions are satisfied.*

3. Melnikov's Method

Equations (1.1) and (1.2) can be written in vector form

$$\frac{d\vec{x}}{dt} = \overrightarrow{f(\vec{x})} + \varepsilon \overrightarrow{g(\vec{x}, t)}, \tag{4}$$

where $\vec{x} = (x, y)$, $\overrightarrow{f(\vec{x})} = (f_1(x, y), f_2(x, y))$, $\overrightarrow{g(\vec{x}, t)} = (g_1(x, y, t), \;\; g_2(x, y, t))$, and $\varepsilon = (\varepsilon_1, \varepsilon_2)$. Equation (4) describes a dynamic system.

Let $J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. If there is a function $H(\vec{x})$ in a system such that $\overrightarrow{f(\vec{x})} = J_2 \bullet \nabla H(\vec{x})$, the system is called as a *generalized Hamiltonian system*. If $H(\vec{x})$ does not exist for the system, it is called as a *non-Hamiltonian system*. When $\varepsilon = 0$, the system is an unperturbed one.

If point $\vec{x} = (x_0, y_0)$ is a solution to the algebraic equation $\overrightarrow{f(\vec{x})} = 0$, then $\vec{x} = (x_0, y_0)$ is called as a *singular point*. From $t = -\infty$ to $t = +\infty$, if (x, y) of $\vec{x}$ satisfies Equation (4), the curve described by (x, y) is called an *orbit* of Equation (4). If an orbit enters the same singular point, when $t = +\infty$ and $t = -\infty$, then the orbit is called a *homoclinic orbit*. If an orbit enters different singular points, respectively, when $t = +\infty$ and $t = -\infty$, then the orbit is called *heteroclinic orbit*.

When $\varepsilon = 0$, for an unperturbed Hamiltonian system, we have [11, 15, 16]

$$\frac{d\vec{x}}{dt} = J_2 \bullet \nabla H(\vec{x}). \tag{5}$$

If the system has a homoclinic orbit $\overrightarrow{q^0(t)} = (x_0(t), y_0(t))^T$ and a hyperbolic saddle point $\overrightarrow{P_0}$, we define its Melnikov's function as

$$M(t_0) \equiv \int_{-\infty}^{+\infty} \vec{f}(\overrightarrow{q^0(t)}) \wedge \vec{g}(\overrightarrow{q^0}(t), t + t_0)\, dt, \tag{6}$$

where $\wedge$ stands for the exterior product.

Recall that the Smale horseshoe map is a strong form of chaos in the mathematics of chaos theory, we have the following theorem.

Theorem 2. *If* $M_i(t_0) = 0$ *has simple zero points, then Equation* (4) *is a chaotic system in the sense of Smale horseshoe.*

When $\varepsilon = 0$, if there are n hyperbolic saddle points $\overrightarrow{p_1}, \overrightarrow{p_2}, \cdots, \overrightarrow{p_n}$ $(n > 2)$ in Equation (5), and there is a heteroclinic orbit $\overrightarrow{q_i^0(t)}$ connecting from $\overrightarrow{p_i}$ to $\overrightarrow{p_{i+1}} (i = 1, 2, \cdots, n-1)$ and a heteroclinic orbit $\overrightarrow{q_n^0(t)}$ connecting from $\overrightarrow{p_n}$ to $\overrightarrow{p_1}$, then we define the Melnikov's function for the i-th heteroclinic orbit as (similar to the homoclinic orbit case).

$$M_i(t_0) \equiv \int_{-\infty}^{+\infty} \vec{f}(\overrightarrow{q_i^0(t)}) \wedge \vec{g}(\overrightarrow{q_i^0}(t), t + t_0)\, dt. \tag{7}$$

Theorem 3. *If* $M_i(t_0) = 0$ *has simple zero points, then Equation* (4) *is a chaotic system in the sense of Smale horseshoe.*

Next, we consider a non-Hamiltonian system

$$\frac{d\vec{x}}{dt} = \overrightarrow{F(\vec{x})} + \varepsilon \overrightarrow{G(\vec{x}, t)}, \tag{8}$$

where $\overrightarrow{F(\vec{x})} = (F_1(\vec{x}),\ F_2(\vec{x}))^T$ is not the vectorial field produced by a Hamiltonian function, and $\overrightarrow{G(\vec{x},\ t)} = (G_1(\vec{x},\ t),\ G_2(\vec{x},\ t))^T \in C^r(r \geq 2)$ is a periodic function of t, in which the period is T, and C^k is a continuous function, that is, k-th order differentiable. When $\varepsilon = 0$, if Equation (8) has a hyperbolic saddle point $\overrightarrow{P^0}$ and a homoclinic orbit $\overrightarrow{Q^0(t)}$ passing through the saddle point, then we have the following theorem.

Theorem 4. *If Melnikov's function*

$$M(t_0) \equiv \int_{-\infty}^{+\infty} \vec{F}(\overrightarrow{Q^0}(t - t_0)) \wedge \vec{G}(\overrightarrow{Q^0}(t - t_0),\ t) \bullet \exp[-\int_0^{t-t_0} Tr\, D\vec{F}(\overrightarrow{Q^0}(\tau)\, d\tau)]\, dt$$

has simple zero points, Equation (8) *is a chaotic system in the sense of Smale horseshoe.*

To see the applications of Melnikov's method, consider the following example of the dynamical system

$$\frac{dx}{dt} = \alpha_1 y, \quad \alpha_1 > 0, \tag{9.1}$$

$$\frac{dy}{dt} = \alpha_2 x - \alpha_3 x^3 + \varepsilon(\gamma \cos \omega t - \delta y), \quad \alpha_2 > 0,\ \alpha_3 > 0, \tag{9.2}$$

where $\alpha_1, \alpha_2, \alpha_3, \gamma,$ and δ are real constants.

The unperturbed system has three singular points (0, 0) and $(\pm\sqrt{\frac{\alpha_2}{\alpha_3}},\ 0)$. The derivative matrix of the singular points (0, 0) is

$$D\vec{f}(0,\ 0) = \begin{pmatrix} 0 & \alpha_1 \\ \alpha_2 & 0 \end{pmatrix}.$$

It has two characteristic values $\lambda_1 = \sqrt{\alpha_1 \alpha_2}$ and $\lambda_2 = -\sqrt{\alpha_1 \alpha_2}$. Because $\lambda_1 > 0 > \lambda_2$, the singular point is a saddle point. It is easy to prove that, the characteristic values of points $\left(\pm\sqrt{\frac{\alpha_2}{\alpha_3}},\, 0\right)$ are a pair of conjugate imaginary numbers. Therefore, points $\left(\pm\sqrt{\frac{\alpha_2}{\alpha_3}},\, 0\right)$ are two centers.

Let's find its Hamiltonian function. From the partial differential equations,

$$\frac{\partial H}{\partial y} = \alpha_1 y,$$

$$-\frac{\partial H}{\partial x} = \alpha_2 x - \alpha_3 x^3,$$

we have

$$H = \frac{1}{2}\alpha_1 y^2 - \frac{1}{2}\alpha_2 x^2 + \frac{1}{4}\alpha_3 x^4.$$

When $\varepsilon = 0$, the system has a conserved quantity

$$\frac{1}{2}\alpha_1 y^2 - \frac{1}{2}\alpha_2 x^2 + \frac{1}{4}\alpha_3 x^4 = c.$$

Because the orbit passes through the origin (0, 0), we have $c = 0$. It can be shown

$$y = \pm\sqrt{\frac{\alpha_2}{\alpha_1}}\,x\sqrt{1 - \frac{\alpha_3}{2\alpha_2}x^2} = \pm a_1 x\sqrt{1 - bx^2}, \tag{10}$$

where $a_1 = \sqrt{\frac{\alpha_2}{\alpha_1}}$ and $b = \frac{\alpha_3}{2\alpha_2}$. From Equation (10), there are two homoclinic orbits. From Equation (9.1), we have

$$\frac{dx}{dt} = \pm ax\sqrt{1 - bx^2}, \tag{10a}$$

where $a = \alpha_1 a_1 = \sqrt{\alpha_1\alpha_2}$.

Solving Equation (10a) and by using the initial conditions, we obtain the two homoclinic orbits: (see Appendix A)

$$\overrightarrow{q_+^0}[x_+^0(t),\ y_+^0(t)] = [\frac{1}{\sqrt{b}}\operatorname{sech}(at),\ -\frac{a_1}{\sqrt{b}}\operatorname{sech}(at)\tanh(at)],$$

$$\overrightarrow{q_-^0}[x_-^0(t),\ y_-^0(t)] = [-\frac{1}{\sqrt{b}}\operatorname{sech}(at),\ +\frac{a_1}{\sqrt{b}}\operatorname{sech}(at)\tanh(at)].$$

Clearly, we have

$$\vec{f}(x,\ y) = (\alpha_1 y_1,\ \alpha_2 x - \alpha_3 x^3)^T,$$

$$\vec{g}(x,\ y) = (0,\ \gamma\cos\omega t - \delta y)^T.$$

According to Equation (6), the Melnikov's function for the homoclinic orbit $\overrightarrow{q_+^0}$ is

$$M_+(t_0) \equiv \int_{-\infty}^{+\infty} \vec{f}(\overrightarrow{q^0}(t)) \wedge \vec{g}(\overrightarrow{q^0}(t),\ t + t_0)\,dt$$

$$= \alpha_1 \int_{-\infty}^{+\infty} y_+^0(t)[\gamma\cos\omega(t + t_0) - \delta y_+^0(t)]$$

$$= \alpha_1 \int_{-\infty}^{+\infty} [-\frac{a_1}{\sqrt{b}}\gamma\ \operatorname{sech}(at)\tanh(at)\cos\omega(t + t_0)$$

$$-\frac{a_1^2}{b}\delta\ \operatorname{sech}^2(at)\tanh^2(at)]\,dt$$

$$= -\frac{2\alpha_1 a_1^2}{3b}\delta + \frac{1}{\sqrt{ba}}\gamma\pi\omega\operatorname{sech}(\frac{\pi\omega}{2a})\sin\omega t_0$$

$$= -I_1 + \Phi\sin\omega t_0,$$

where $I_1 = \frac{2\alpha_1 a_1^2}{3b}\delta,\ \Phi = \frac{1}{\sqrt{ba}}\gamma\pi\omega\operatorname{sech}(\frac{\pi\omega}{2a})$.

If $M_+(t_0) = 0,$ then we have

$$\sin\omega t_0 = \frac{I_1}{\Phi}. \tag{11a}$$

Only when $|I_1| < |\Phi|$, Equation (11a) has solutions. If t_0^* is a solution to Equation (11a), then

$$M_+(t_0^*) = -I_1 + \Phi \sin \omega t_0^* = 0.$$

We have

$$\frac{d}{dt_0} M_+(t_0^*) = -\Phi\omega \cos \omega t_0^* \neq 0.$$

So, t_0^* is a simple zero point.

Clearly, from Theorem 2, when the following condition

$$\left| \frac{\gamma\sqrt{\alpha_3}}{\sqrt{2}\alpha_2^{\frac{3}{2}}} \omega \operatorname{sech}\left(\frac{\pi\omega}{2\sqrt{\alpha_1\alpha_2}} \right) \right| > \frac{2}{3\pi} |\delta| \tag{11}$$

is satisfied, the system is chaotic in the sense of Smale horseshoe.

Let $\Phi_1 = \frac{\gamma\sqrt{\alpha_3}}{\sqrt{2}\alpha_2^{\frac{3}{2}}} \omega \operatorname{sech}\left(\frac{\pi\omega}{2\sqrt{\alpha_1\alpha_2}} \right)$. It can be proven that Φ_1 is an increasing function of α_1 and α_3 and when Equation (11b) is satisfied, $\frac{\partial\Phi_1}{\partial\alpha_2} > 0$. In fact, when

$$\frac{\partial\Phi_1}{\partial\alpha_3} = \frac{\gamma\omega}{2\sqrt{2}\sqrt{\alpha_3}\alpha_2^{\frac{3}{2}}} \operatorname{sech}\left(\frac{\pi\omega}{2\sqrt{\alpha_1}\sqrt{\alpha_2}} \right) > 0,$$

$$\frac{\partial\Phi_1}{\partial\alpha_1} = \frac{\gamma\omega\sqrt{\alpha_3}}{\sqrt{2}\alpha_2^{\frac{3}{2}}} \frac{\partial}{\partial\alpha_1} \operatorname{sech}\left(\frac{\pi\omega}{2\sqrt{\alpha_1}\sqrt{\alpha_2}} \right)$$

$$= \frac{\pi\gamma\omega^2\sqrt{\alpha_3}}{4\sqrt{2}\alpha_1^{\frac{3}{2}}\alpha_2^2} \operatorname{sech}\left(\frac{\pi\omega}{2\sqrt{\alpha_1}\sqrt{\alpha_2}} \right) \tanh\left(\frac{\pi\omega}{2\sqrt{\alpha_1}\sqrt{\alpha_2}} \right) > 0,$$

$$\frac{\partial\Phi_1}{\partial\alpha_2} = \frac{\gamma\omega\sqrt{\alpha_3}}{\sqrt{2}} \frac{\partial}{\partial\alpha_2} \left[\alpha_2^{\frac{3}{2}} \operatorname{sech}\left(\frac{\pi\omega}{2\sqrt{\alpha_1}\sqrt{\alpha_2}} \right)\right]$$

$$= \frac{\gamma\omega\sqrt{\alpha_3}}{2\sqrt{2}\alpha_2^{\frac{5}{2}}} \operatorname{sech}\left(\frac{\pi\omega}{2\sqrt{\alpha_1}\sqrt{\alpha_2}}\right)\left[\frac{\pi\omega}{2\sqrt{\alpha_1}\sqrt{\alpha_2}} \tanh\left(\frac{\pi\omega}{2\sqrt{\alpha_1}\sqrt{\alpha_2}}\right) - 3\right].$$

When

$$\frac{\pi\omega}{\sqrt{\alpha_1}\sqrt{\alpha_2}} \tanh\left(\frac{\pi\omega}{2\sqrt{\alpha_1}\sqrt{\alpha_2}}\right) > 6, \tag{11b}$$

we have $\frac{\partial \Phi_1}{\partial \alpha_2} > 0$. Therefore, the chaotic condition, Equation (11), is more likely to hold with larger values of α_1 and α_2.

4. Application to Chinese Stock Markets

In contrast to stock markets in the developed countries and areas, Chinese stock markets have the following characteristics.

4.1. A large number of individual investors and an extensive amount of capital

Chinese stock markets are dominated by individual investors, who have a huge amount of capital in hand, and are generally unsophisticated and short-term oriented. As a result, there are extreme movements in the demand for stocks and thus high volatility in Chinese stock markets. The statistics from China Securities Depository and Clearing Corporation shows that as of May 28, 2007, Chinese investors opened 87.25 million investment accounts, the majority of which is held by individual investors [35]. One of the official websites, www.cnstock.com, reports that individual investor's trading account for as high as 60.1% of the volume in Shanghai Stock Exchanges for the first three months of 2007; 64.7% of market shares in Shenzhen Stock Exchanges is held by individual investors [36]. These individual investors hold an extensive amount of capital. According to the People's Bank of China, the total saving of Chinese households was over US$2.5 trillion at the end of 2007, which was twice the total market worth of *A*-share [37]. The dominance of Chinese individual investors and their unsophistication highlight the speculative nature of Chinese stock markets, which is likely to cause large movements in the demand of stock shares.

4.2. Different classes of stock and non-circulating shares

Companies listed in Chinese stock markets have different classes of stocks: state-owned shares, legal person shares, domestic listed shares (*A*-shares are traded in Chinese Yuan and are available to domestic investors, and *B*-shares are traded in foreign currencies and are available only to foreign investors until February 19, 2001), and shares listed abroad (e.g., *H* shares are listed in Hong Kong, and *N* shares are listed in New York). The state-owned shares and legal person shares account for about two thirds of the total shares. The two classes of shares are non-circulating meaning that, they are not publicly traded except by special approval of the central and local governments. Generally, there are two types of non-circulating shares: large non-circulating shares (LN) refer to the amount of shares that are more than 5% of the total shares of the listed company, and small non-circulating shares (SN) refer to the amount of shares that are less than 5% of the total shares. Shanghai Stock Exchange and Shenzhen Stock Exchange started the stock reform in 2005. LN and SN are permitted to decrease their holding shares step by step according to a define rule. Non-circulating shareholders took the advantage of the stock reform, and sold their stock ownership for easy capital gains. The trading of SN and LN on Chinese stock markets has greatly increased the amount of shares available in the markets. The speculative incentives of non-circulating shareholders are likely to lead to extreme movements in the supply of stocks, and thus high volatility in Chinese stock markets.

4.3. High sensitivity to government regulations

Chinese stock markets are very sensitive to government regulations. Su and Fleisher find that large volatility hikes in Chinese stock markets are associated with exogenous changes in government regulations [30]. For example, China Securities Regulatory Commission (hereafter CSRC) removed a 5% limit on daily stock price movements on May 21, 1992, when the index of Shanghai Stock Exchange *A*-share doubled in one day. In order to restrain volatility, CSRC re-imposed a 10% daily price-change limit on individual stocks on December 13, 1996. Not surprisingly, the regulation change was followed by 5 consecutive days of decrease in the

Shanghai Stock Exchange index. In addition, CSRC relaxed the restriction on the trading of non-circulating shares in August 2006, which was widely believed to contribute to the massive amount of the index decline starting from November 2007. Chinese stock markets have a relatively short history-they were only re-opened in early 1990s after being closed for nearly half a century. Therefore, the Chinese government is still going through the learning process and lack of continuous and consistent policies. This again may lead to high volatility in Chinese stock markets.

4.4. The speculative incentives of domestic investors

The speculative incentives of domestic investors create a speculative component of stock prices, especially for A-share. Prior studies document that severe misevaluation of A-shares in Chinese stock markets is in part due to the joint effect of short-sale prohibition, and investors' heterogeneous beliefs on stock value [14, 21, 23, 24]. Specifically, stock prices will be biased upward, if there is a sufficient amount of divergent opinions, since the marginal buyer of shares is likely among optimists (pessimists, who are prevented from short-selling, simply sit out of the market). Because of the upward bias in stock prices, investors could be willing to pay more than their expected stock value, anticipating that they could sell the shares to other investors, who pay even more in the future. The existence of the speculative component of stock prices may result in extreme movements of equity values in Chinese stock markets.

The above characteristics shed lights on the applications of our model to Chinese stock markets. In the logistic model, we have

$$\frac{\partial x}{\partial t} = Ax - Bx^2, \quad \mu = A + 1, \tag{12}$$

where $A = -(kN)_0$. When $N_d > N_s$, we have $N = N_s - N_d < 0$ and $A > 0$. If k is big enough, it can be true that $\mu > \mu_\infty$. In other words, a large A suggests that the increase in stock price over time is more sensitive to the current price, when there is increasing buying pressure due to demand surplus. The quadratic term Bx^2, which is a necessary condition for a chaotic system, indicates that the change in stock price over time is a nonlinear function of the current price, and the decrease in price is greater, if selling pressure due to supply surplus is greater. Given that there are extreme movements in the supply and demand of stocks, and Chinese stock markets are highly sensitive to government regulations and full of speculative investors, the changes in stock price over time is likely nonlinear, and the market may be chaotic.

As for the Melnikov's method, there is a quadratic term $-\alpha_3 x^3$ in Equation (9.2). Because Φ_1 is an increasing function of α_1, α_2, and α_3, when Equation (11b) is satisfied, large values of α_1, α_2, and α_3 are likely to result in chaos. The larger the values of α_1, α_2, and α_3 are, the higher the sensitivity of the market's reactions to the current price. There is a periodic perturbation term $\gamma \cos \omega t$ in Equation (9.2). Given that stock prices go up and down, the change of stock price may generate periodicity to a certain extent. Besides the periodic perturbation term, there may be a non-periodic term in $\vec{g}(x, y, t)$, and we have also included a correction term δy. Considering the unique features of Chinese stock markets earlier discussed, the above conditions are easily satisfied, and thus, the markets are more likely to be chaotic. As a result, our model can describe the chaotic characteristics of Chinese stock markets.

The predictions of our model are in line with previous empirical evidence on Chinese stock markets. Peters [28] points out two characteristics are essential for a chaotic system: the fractal dimension and the sensitivity to initial conditions, and provides evidence that most of the capital markets are fractal. The chaotic behavior of Chinese stock markets has been documented by a group of papers. For example, by using the high-frequency data of the Shanghai Stock Exchange Composite Index (hereafter SSECI) in 2002, Wang et al. analyze its nonlinear characteristics and found that the Lyapunov index is positive. This is consistent with the chaotic behavior of the SSECI [32]. Using three different methods of multifractal analysis, Du and Ning [4] examine the variation in the SSECI and found evidence that the index has multifractal features. More importantly, they show that, as the order of the partition function increases, the multifractal strength increases, the singular spectrums become tougher, and the general Hurst exponents decrease. In addition, Zhuang et al. [34] show that, the indices of both Shanghai Stock Exchange and Shenzhen Stock Exchange do not follow the normal distribution, but rather exhibit fractal time series.

To illustrate that the multifractal features of Chinese stock markets by using our model, take SSECI, for example. The base value of the SSECI is 100, which is for the base day December 19, 1990. The SSECI for day t is calculated as

$$Index_t = 100 \times \frac{\sum_{i=1}^{n_t} x_{t,i} N_{st,i}}{\sum_{i=1}^{n_0} x_{0,i} N_{s0,i}}, \tag{13}$$

where $x_{0,i}$ and $N_{0,i}$ are the price and the number of shares issued for the stock i on the base day, respectively, $x_{t,i}$ and $N_{t,i}$ are the price and the number of shares issued for the stock i on day t, and n_0 and n_t are the number of stocks on the base day and day t. The time series of prices for each stock has its own scaling characteristics, i.e., fractal dimension, and different time series of stock prices can have different scaling characteristics. Since the time series of the SSECI summarizes price fluctuations of all n stocks, it is more appropriate to describe the index by using a multi-scale fractal structure. In terms of our model, we have constructed a spectrum $f(d)$ by using an infinite sequence of different fractal dimensions d_i. $f(d)$, which is called a *singular spectrum*, has been employed by prior studies to describe the characteristics of a multi-scale fractal structure. We allow $f_1(x, y)$, $f_2(x, y)$, $g_1(x, y, t)$, and $g_2(x, y, t)$ in Equations (1.1) and (1.2) to take different forms for different stocks. By doing so, we believe that our model have properly accounted for the multifractal characteristics of the SSECI.

5. Conclusion

Based on our model as well as previous empirical evidence on Chinese stock markets, we draw following conclusions.

(1) By adjusting $\vec{f}(\vec{x})$ and $\vec{g}(\vec{x}, t)$ according to the status of stock markets, Equation (4) can be used to describe stock price changes. If certain conditions are satisfied, then the stock market is chaotic; if the conditions are not satisfied, the stock market is not chaotic. In other words, a stock market can fall into or rise out of chaos. An important characteristic of chaos is high sensitivity to initial conditions. Since a stock market is not always sensitive to initial conditions, the market can be efficient for a certain period.

(2) Generally speaking, our model is descriptive of all stock markets, but it is especially so for Chinese stock markets, due to the unique features of the markets. Said differently, Chinese stock markets can easily fall into chaos.

(3) The different forms of $\vec{f}(\vec{x})$ and $\vec{g}(\vec{x}, t)$ can affect the status of a stock market to a great extent. For example, if $\alpha_2 = -\beta_2 < 0$ and $\alpha_3 = -\beta_3 < 0$ in Equation (9.2), the equation becomes

$$\frac{dx}{dt} = \alpha_1 y, \tag{14.1}$$

$$\frac{dy}{dt} = -\beta_2 x + \beta_3 x^3 + \varepsilon(f \cos \omega t - \delta y), \tag{14.2}$$

where $\beta_2 > 0$ and $\beta_3 > 0.$ When $\varepsilon = 0,$ the unperturbed equations are

$$\frac{dx}{dt} = \alpha_1 y, \tag{15.1}$$

$$\frac{dy}{dt} = -\beta_2 x + \beta_3 x^3. \tag{15.2}$$

The unperturbed system has three singular points (0, 0) and $(\pm\sqrt{\frac{\beta_2}{\beta_3}},\ 0)$. It is easy to show that point (0, 0) is a center and points $(\pm\sqrt{\frac{\beta_2}{\beta_3}},\ 0)$ are saddle points. There are two heteroclinic orbits connecting the two saddle points.

It can be proven that, when

$$\left| \frac{f\sqrt{\beta_3}}{\beta_2^{\frac{3}{2}}}\,\omega\ \mathrm{csch}\left(\frac{\pi\omega}{\sqrt{2}\sqrt{\alpha_1\beta_2}}\right) \right| > \frac{2}{3\pi}\,|\,\delta\,|, \tag{16}$$

the system is chaotic in the sense of Smale horseshoe (see Appendix B).

$$\text{Let } \Phi_2 = \frac{f\sqrt{\beta_3}}{\beta_2^{\frac{3}{2}}}\,\omega\ \mathrm{csch}\left(\frac{\pi\omega}{\sqrt{2}\sqrt{\alpha_1\beta_2}}\right). \tag{16a}$$

It can be shown that Φ_2 is an increasing function of α_1 and β_3. When certain conditions are satisfied, Φ_2 is also an increasing function of β_2. Therefore, large values of α_1 and β_3 are likely to result in chaos. The above is consistent with the notion that high sensitivity is a necessary condition of chaos.

(4) Our model can be capable of describing the highly speculative nature of Chinese stock markets. Because $\vec{f}(\vec{x})$ and $\vec{g}(\vec{x},\ t)$ are affected by multiple factors, the two functions can take various forms, and thus, Chinese stock markets are more likely to have large movements unexplained by a linear paradigm. For example, the SSECI was down 71.8% to 1,720 as of October 29, 2008 from its record close of 6,092 on October 16, 2007. The decline in the SSECI was quite dramatic, comparing to the country’s officially recorded GDP growth rates of 11.4% in 2007 versus 9% in 2008. Thus, such turbulent movements in stock prices, which might be largely caused by speculations, manifest the legitimacy of application the chaotic model to Chinese stock prices.

Appendix A

From Equation (10a), we have

$$t = \pm \int \frac{dx}{ax\sqrt{1 - bx^2}}. \tag{A1}$$

Upon integration of Equation (A1), we obtain

$$t = \pm \frac{1}{a}\left[\log(\sqrt{b}x) - \log(1 + \sqrt{1 - bx})\right]. \tag{A2}$$

From Equation (A2), we obtain

$$x = \frac{1}{\sqrt{b}} \operatorname{sech}(at) + c_1. \tag{A3}$$

If $t = \pm\infty$, then $x = 0$, and thus, we have $c_1 = 0$ (see Figure 1).

Substituting Equation (A3) into Equation (10), we have

$$y = \pm \frac{a_1}{\sqrt{b}} \operatorname{sech}(at)\tanh(at) + c_2.$$

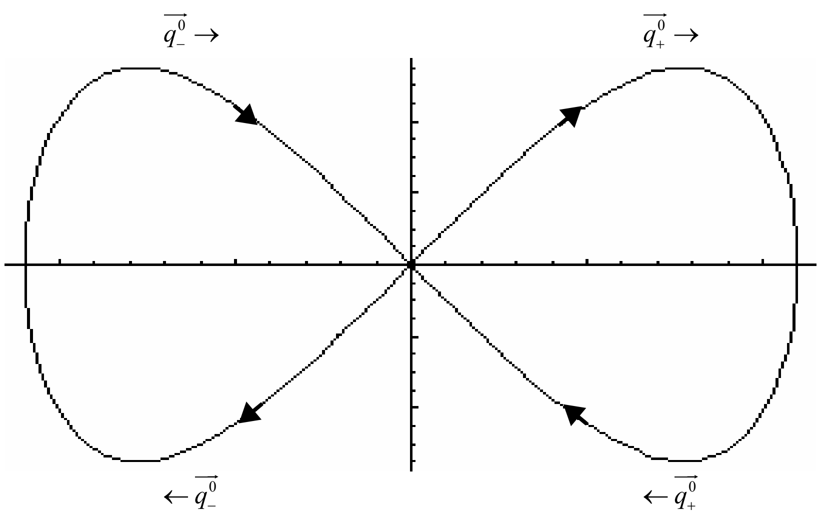

Figure 1. Two homoclinic orbits.

When $t = \pm\infty$, $y = 0$, so $c_2 = 0$. From Figure 1, when t is in the interval $(-\infty, 0)$, $y > 0$ for $\overrightarrow{q_+^0}$, so we have

$$\overrightarrow{q_+^0}[x_+^0(t),\ y_+^0(t)] = [\frac{1}{\sqrt{b}}\operatorname{sech}(at),\ -\frac{a_1}{\sqrt{b}}\operatorname{sech}(at)\tanh(at)],$$

$$\overrightarrow{q_-^0}[x_-^0(t),\ y_-^0(t)] = [-\frac{1}{\sqrt{b}}\operatorname{sech}(at),\ +\frac{a_1}{\sqrt{b}}\operatorname{sech}(at)\tanh(at)].$$

$$M(t_0) = I_1 + I_2,$$

$$I_1 = \frac{\alpha_1 a_1^2 \delta}{b}\int_{-\infty}^{+\infty}\operatorname{sech}^2(at)\tanh^2(at)\,dt = \frac{2}{3}\,\frac{\alpha_1 a_1^2\delta}{ba}.$$

Noticing

$$\cos\omega(t+t_0) = \cos\omega t\cos\omega t_0 - \sin\omega t\sin\omega t_0,$$

we have

$$I_2 = -\alpha_1\int_{-\infty}^{+\infty}\frac{a_1\gamma}{\sqrt{b}}\operatorname{sech}(at)\tanh(at)\cos\omega(t+t_0)\,dt$$

$$= \frac{\alpha_1 a_1\gamma}{\sqrt{b}}\sin\omega t_0\int_{-\infty}^{+\infty}\operatorname{sech}(at)\tanh(at)\sin\omega t dt$$

$$= \frac{2\alpha_1 a_1\gamma}{\sqrt{b}}\sin\omega t_0\int_{0}^{+\infty}\operatorname{sech}(at)\tanh(at)\sin\omega t dt$$

$$= -\frac{2\gamma}{\sqrt{b}}\sin\omega t_0\int_{0}^{+\infty}\sin\omega t d\operatorname{sech}(at)$$

$$= \frac{2\gamma}{\sqrt{b}}\sin\omega t_0\int_{0}^{+\infty}\operatorname{sech}(at) d\sin\omega t$$

$$= \frac{\gamma\omega\pi}{a\sqrt{b}}\sin\omega t_0\operatorname{sech}(\frac{\pi\omega}{2a}). \tag{A4}$$

Appendix B

From Equations (15.1) and (15.2), we obtain the derivation matrix of the singular point (0, 0), which is

$$D\vec{f}(0,\ 0) = \begin{pmatrix} 0 & \alpha_1 \\ -\beta_2 & 0 \end{pmatrix}.$$

The characteristic values are $\lambda = \pm\sqrt{-\alpha_1\beta_2}$. This is a pair of conjugate imaginary numbers, so point (0, 0) is a center.

The derivation matrix of the singular point $\left(\pm\sqrt{\frac{\beta_2}{\beta_3}},\ 0\right)$ is

$$D\vec{f}\left(\pm\sqrt{\frac{\beta_2}{\beta_3}},\ 0\right) = \begin{pmatrix} 0 & \alpha_1 \\ 2\beta_2 & 0 \end{pmatrix}.$$

The characteristic values are $\lambda = \pm\sqrt{2\alpha_1\beta_2}$. Because $\lambda_1 = \sqrt{2\alpha_1\beta_2} > 0 > \lambda_2 = -\sqrt{2\alpha_1\beta_2}$, $\left(\pm\sqrt{\frac{\beta_2}{\beta_3}},\ 0\right)$ are saddle points.

From Equations (15.1) and (15.2), we obtain the Hamiltonian function of the system, which is

$$H = \frac{1}{2}\alpha_1 y^2 + \frac{1}{2}\beta_2 x^2 - \frac{1}{4}\beta_3 x^4.$$

When $\varepsilon = 0$, the system has a conserved quantity

$$\frac{1}{2}\alpha_1 y^2 + \frac{1}{2}\beta_2 x^2 - \frac{1}{4}\beta_3 x^4 = c_3. \tag{A5}$$

The orbits pass through points $\left(\pm\sqrt{\frac{\beta_2}{\beta_3}},\ 0\right)$, so we have $c_3 = \frac{1}{4}\frac{\beta_2^2}{\beta_3}$.

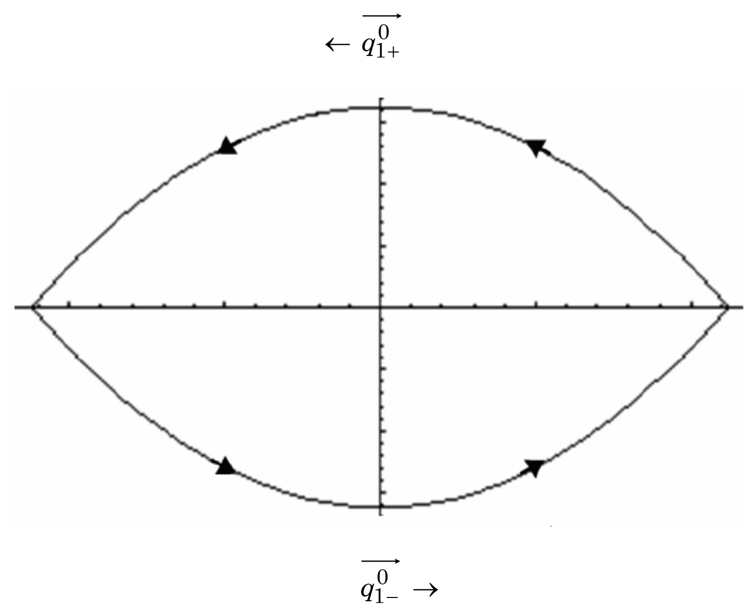

Figure 2. Two heteroclinic orbits.

From Equation (A5), we obtain

$$y = \pm a_2(b_1 x^2 - 1), \tag{A6}$$

where $a_2 = \dfrac{\beta_2}{\sqrt{2\alpha_1\beta_3}}$ and $b_1 = \dfrac{\beta_3}{\beta_2}$.

From Equations (15.1) and (A6), we obtain

$$\frac{dx}{dt} = \pm a'(b_1 x^2 - 1), \tag{A7}$$

where $a' = \sqrt{\dfrac{\alpha_1}{2\beta_3}}\beta_2$. From Equation (A7), we have

$$dt = \pm \frac{dx}{a'(b_1 x^2 - 1)}. \tag{A8}$$

Upon integrating of Equation (A8), we obtain

$$t = \mp \frac{1}{a'\sqrt{b_1}} \, Arc\tanh(\sqrt{b_1}\, x). \tag{A9}$$

From Equation (A9), we obtain

$$x = \mp \frac{1}{\sqrt{b_1}} \tanh(a'\sqrt{b_1}\, t) + c_4. \tag{A10}$$

For $\overrightarrow{q_{1+}^0}$, when $t = \pm\infty$, $x = \mp\sqrt{\frac{\beta_2}{\beta_3}} = \mp\frac{1}{\sqrt{b_1}}$, so we have $c_4 = 0$.

Substituting Equation (A10) into Equation (A6), we have

$$y = \pm a_2 \operatorname{sech}^2(a'\sqrt{b_1}\, t). \tag{A11}$$

When t is in the interval $(-\infty, +\infty)$, $y_{1+} > 0$ for $\overrightarrow{q_{1+}^0}$ (see Figure 2). We obtain two heteroclinic orbits

$$\overrightarrow{q_{1+}^0}[x_{1+}^0(t),\ y_{1+}^0(t)] = \left[\mp \frac{1}{\sqrt{b_1}} \tanh(a'\sqrt{b_1}\, t),\ a_2 \operatorname{sech}^2(a'\sqrt{b_1}\, t)\right],$$

$$\overrightarrow{q_{1-}^0}[x_{1-}^0(t),\ y_{1-}^0(t)] = \left[\pm \frac{1}{\sqrt{b_1}} \tanh(a'\sqrt{b_1}\, t),\ -a_2 \operatorname{sech}^2(a'\sqrt{b_1}\, t)\right].$$

From Equation (7), we have

$$\begin{aligned}
M_+(t_0) &= \int_{-\infty}^{+\infty} \vec{f}(\overrightarrow{q_{1+}^0}(t)) \wedge \vec{g}(\overrightarrow{q_{1+}^0}(t),\ t + t_0)\, dt \\
&= \int_{-\infty}^{+\infty} \begin{pmatrix} \alpha_1 y_{1+}^0(t) \\ -\beta_2 x_{1+}^0(t) + \beta_3 [x_{1+}^0(t)]^3 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ -\delta y_{1+}^0(t) + f\cos\omega(t + t_0) \end{pmatrix} dt \\
&= \int_{-\infty}^{+\infty} [-\delta y_{1+}^0(t) + f\cos\omega(t + t_0)]\alpha_1 y_{1+}^0(t)\, dt \\
&= -\alpha_1 a_2^2 \delta \int_{-\infty}^{+\infty} \operatorname{sech}^4(a'\sqrt{b_1}\, t)\, dt
\end{aligned}$$

$$+ \alpha_1 f a_2 \int_{-\infty}^{+\infty} \operatorname{sech}^2(a'\sqrt{b_1}t)\cos\omega(t+t_0)\,dt = I_3 + I_4.$$

$$I_3 = -\alpha_1 a_2^2\,\delta \int_{-\infty}^{+\infty} \operatorname{sech}^4(a'\sqrt{b_1}t)\,dt$$

$$= -\frac{\alpha_1 \delta a_2^2}{a'\sqrt{b_1}} \int_{-\infty}^{+\infty} \operatorname{sech}^4(t')\,dt' = -\frac{4}{3}\frac{\alpha_1 a_2^2}{a'\sqrt{b_1}}\delta.$$

Noticing

$$\cos\omega(t+t_0) = \cos\omega t\cos\omega t_0 - \sin\omega t\sin\omega t_0,$$

we have

$$I_4 = \alpha_1 f a_2 \cos\omega t_0 \int_{-\infty}^{+\infty} \operatorname{sech}^2(a'\sqrt{b_1}t)\cos\omega t dt$$

$$= \frac{2\alpha_1 f a_2}{a'\sqrt{b_1}}\cos\omega t_0 \int_{0}^{+\infty} \operatorname{sech}^2(t')\cos\frac{\omega t'}{a'\sqrt{b_1}}\,dt'$$

$$= \frac{\alpha_1 f a_2 \pi\omega}{a'^2 b_1}\cos\omega t_0\ \operatorname{csch}\left(\frac{\pi\omega}{2a'\sqrt{b_1}}\right).$$

The condition for $M_+(t_0) = 0$ is

$$\left|\frac{f\omega}{a_2 a'\sqrt{b_1}}\operatorname{csch}\left(\frac{\pi\omega}{2a'\sqrt{b_1}}\right)\right| > \frac{4}{3\pi}|\,\delta\,|.$$

From $\Phi_2 = \dfrac{f\sqrt{\beta_3}}{\beta_2^{\frac{3}{2}}}\,\omega\operatorname{csch}\left(\dfrac{\pi\omega}{\sqrt{2}\sqrt{\alpha_1\beta_2}}\right),$ (16a)

we have

$$\frac{\partial\Phi_2}{\partial\beta_3} = \frac{f}{2\sqrt{\beta_3}\beta_2^{\frac{3}{2}}}\,\omega\operatorname{csch}\left(\frac{\pi\omega}{\sqrt{2}\sqrt{\alpha_1\beta_2}}\right) > 0,$$

$$\frac{\partial \Phi_2}{\partial \alpha_1} = \frac{f \pi \omega^2 \sqrt{\beta_3}}{2\sqrt{2}\alpha_1^{\frac{3}{2}} \beta_2^2} \coth\left(\frac{\pi \omega}{\sqrt{2}\sqrt{\alpha_1 \beta_2}}\right) \operatorname{csch}\left(\frac{\pi \omega}{\sqrt{2}\sqrt{\alpha_1 \beta_2}}\right) > 0,$$

$$\frac{\partial \Phi_2}{\partial \beta_2} = \frac{f\, \omega \sqrt{\beta_3}}{2\, \beta_2^{\frac{5}{2}}} \operatorname{csch}\left(\frac{\pi \omega}{\sqrt{2}\sqrt{\alpha_1 \beta_2}}\right) \left[\frac{\pi \omega}{\sqrt{2}\sqrt{\alpha_1 \beta_2}} \coth\left(\frac{\pi \omega}{\sqrt{2}\sqrt{\alpha_1 \beta_2}}\right) - 3\right].$$

When $\frac{\pi \omega}{\sqrt{2}\sqrt{\alpha_1 \beta_2}} \coth\left(\frac{\pi \omega}{\sqrt{2}\sqrt{\alpha_1 \beta_2}}\right) > 3,$ (A12)

we have $\frac{\partial \Phi_2}{\partial \beta_2} > 0.$

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